Fundamentals in Nuclear Physics 原子核基礎

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Elements of

- Special Relativity　　特殊相対性理論
- Quantum Mechanics　　量子力学

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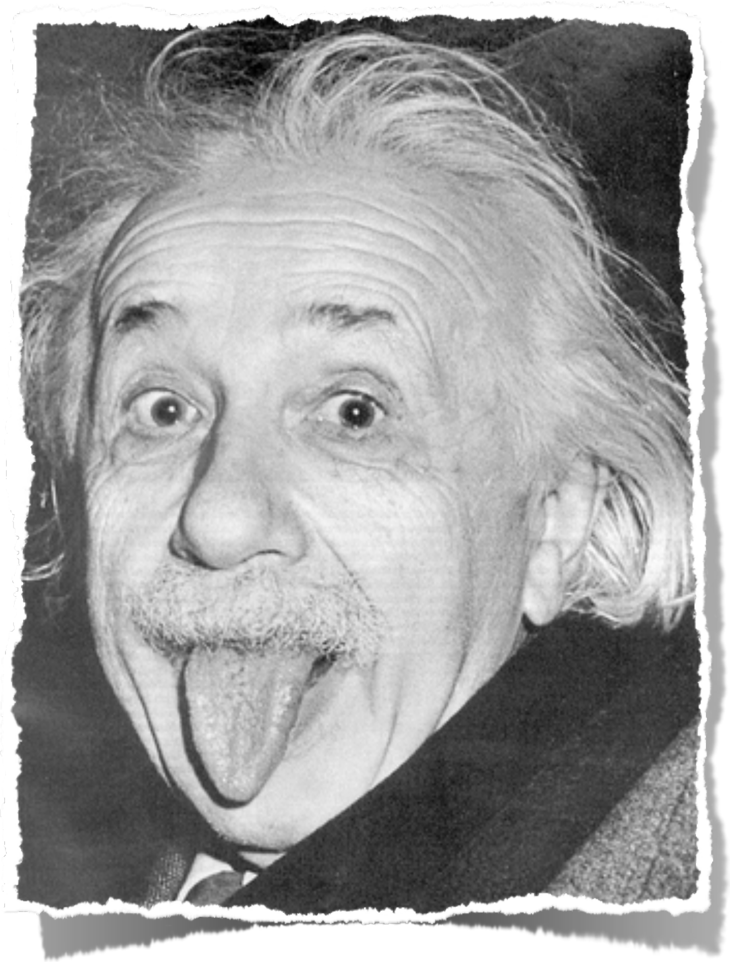
Special relativity

特殊相対性理論



1905

Special relativity is derived from two principles.

① Special Principle of Relativity 特殊相対性原理

Physical laws should be the same in every inertial frame of reference. 慣性系

② Principle of Invariant Light Speed 光速度不変の原理

There is at least one inertial frame of reference where Maxwell's equations hold.

Maxwell's equations

1864年

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = \mathbf{0}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

in vacuum

真空中

$$\frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} - \nabla^2 \mathbf{E} = 0$$

$$c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

vacuum velocity of light is uniquely derived from Maxwell's equations

Special relativity is derived from two principles.

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There is at least one inertial frame of reference where Maxwell's equations hold.

➡ light in vacuum propagates with the speed *c* in *one* inertial frame of reference, regardless of the state of motion of the light source.

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Relativity is used in nuclear physics primarily through the expressions for the energy and momentum of a free particle

運動量

rest mass 静止質量 m velocity 速度 $\mathbf{v}$

energy $$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2 = Mc^2$$

momentum $$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} = \gamma m\mathbf{v} = M\mathbf{v}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} \qquad \beta = v/c$$

rest mass 静止質量 m velocity 速度 $\mathbf{v}$

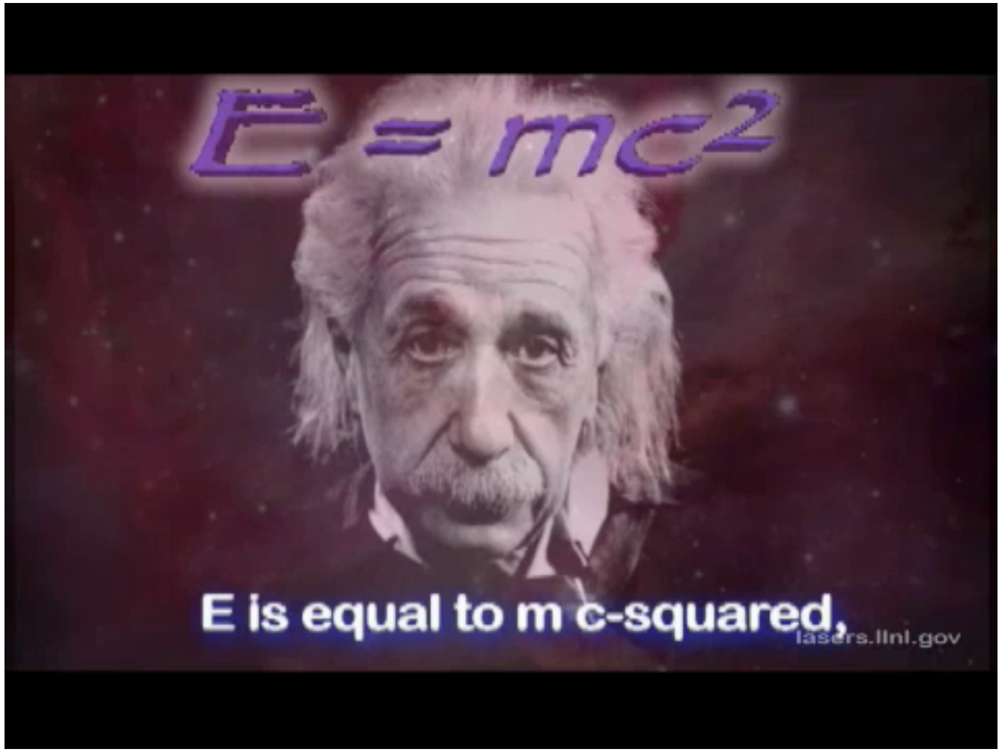
energy $$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2 = Mc^2$$

non-relativistic limit $v \ll c$ usually applies for nuclei

$$E \approx mc^2 + \frac{1}{2}mv^2$$

- Mass is a form of energy

➡ nuclear energy


E = mc2
E is equal to m c-squared,
lasers.llnl.gov

rest mass 静止質量 m velocity 速度 $\mathbf{v}$

energy $$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2 = Mc^2$$

non-relativistic limit $v \ll c$ usually applies for nuclei

$$E \approx mc^2 + \frac{1}{2}mv^2$$

- Mass is a form of energy
- The faster the particle is, the larger its observed mass is

The kinetic energy 運動エネルギー is also observed as a part of the (observed) mass

rest mass 静止質量 m velocity 速度 $\mathbf{v}$

energy $$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} = \gamma mc^2 = Mc^2$$

Any form of energy is observed as mass.

energy $E = \dfrac{mc^2}{\sqrt{1 - v^2/c^2}}$ momentum $\mathbf{p} = \dfrac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}$

$$E^2 = m^2c^4 + p^2c^2 \qquad \frac{v}{c} = \frac{pc}{E}$$

nuclei: non-relativistic limit $v \ll c$

$$E \approx mc^2 + \frac{p^2}{2m} \qquad p \approx mv$$

neutrinos and photons: $mc \ll p$

$$E \approx pc\left(1 + \frac{m^2c^2}{2p^2}\right) \qquad v \approx c\left(1 - \frac{m^2c^2}{2p^2}\right)$$

especially for photons: $m = 0$

$$E = pc \qquad v = c$$

For two particles *A* and *B*

$$E_A E_B - c^2\,\mathbf{p_A}\cdot\mathbf{p_B}$$

is independent of inertial frames of reference (Lorentz invariant)

ローレンツ不変量

Especially $$E^2 - c^2p^2 = m^2c^4$$

Decay $\mathrm{A} \to \mathrm{B} + \mathrm{C}$

find $\mathbf{p}_B, \mathbf{p}_C$

Energy conservation $E_A = E_B + E_C$

Momentum conservation $\mathbf{p}_A = \mathbf{p}_B + \mathbf{p}_C$

In the rest frame of A　Aの静止系で

$$\mathbf{p}_A = 0 \longrightarrow p_B = -p_C$$

$$m_A c^2 = \sqrt{p^2c^2 + m_B^2 c^4} + \sqrt{p^2c^2 + m_C^2 c^4}$$

not easy to solve ...

$$E_C = E_A - E_B \qquad \mathbf{p}_C = \mathbf{p}_A - \mathbf{p}_B$$

$$E_C^2 - c^2\mathbf{p}_C^2 = (E_A - E_B)^2 - c^2(\mathbf{p}_A - \mathbf{p}_B)^2$$

$$m_C^2c^4 = m_A^2c^4 + m_B^2c^4 - 2(E_AE_B - c^2\mathbf{p}_A\cdot\mathbf{p}_B)$$

Lorentz invariant

can be evaluated with a convenient inertial frame of reference

In the rest frame of A

$$m_C^2c^4 = m_A^2c^4 + m_B^2c^4 - 2m_Ac^2\sqrt{m_B^2c^4 + p^2c^2}$$

$$p^2 = \left[\left(\frac{m_A^2 + m_B^2 - m_C^2}{2m_A}\right)^2 - m_B^2\right]c^2$$

Elements of Quantum Mechanics

Wave-particle duality to the Schrödinger equation

粒子波動二重性　　　　シュレーディンガー方程式

plane wave $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$

平面波

$$\mathbf{p} = \hbar\mathbf{k}, \quad E = \hbar\omega$$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34}\,\mathrm{J\cdot s}$$ reduced Planck constant

$$h = 6.626 \times 10^{-34}\,\mathrm{J\cdot s}$$ Planck constant　プランク定数

how to extract *p* and *E* from $\Psi(x,t) = e^{i(kx-\omega t)}$?

$$p\Psi(x,t) = -i\hbar\frac{\partial}{\partial x}\Psi(x,t)$$

$$E\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}, \ \hat{E} = i\hbar\frac{\partial}{\partial t}$$

In quantum mechanics, a physical quantity is represented by an operator (or matrix) in general.

$$E = \frac{p^2}{2m} + V(x)$$

kinetic energy　運動エネルギー

potential energy　ポテンシャルエネルギー

$$\hat{p} = -i\hbar\frac{\partial}{\partial x}, \quad \hat{E} = i\hbar\frac{\partial}{\partial t}$$

波動関数
wave function

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi(x,t)$$

the time-dependent Schrödinger equation (1D)

時間に依存するシュレーディンガー方程式

3D

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(x)\right]\Psi(\mathbf{r},t)$$

$|\Psi(x,t)|^2$ or $|\Psi(\mathbf{r},t)|^2$ interpreted as probability density to find the particle at *x*

Solution with energy $E = \hbar\omega$

$$\Psi(x,t) = \psi(x)e^{-i\omega t}$$

$$\Rightarrow \boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi(x) = E\,\psi(x)}$$

the (time-independent) Schrödinger equation (1D)

E : energy eigenvalue　エネルギー固有値

波動関数
wave function

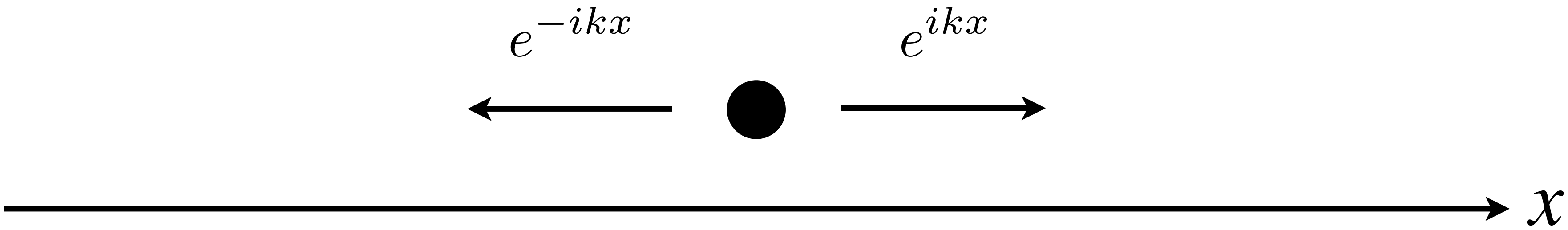
$\psi(x)$ and $\frac{d\psi}{dx}$ continuous functions
連続関数

Free particle 自由粒子 $V(x) = 0$

$$\frac{d^2\psi}{dx^2} + \frac{2mE}{\hbar^2}\psi(x) = 0 \quad \Rightarrow \quad \psi(x) = e^{ikx}, \quad e^{-ikx}$$

平面波
plane wave

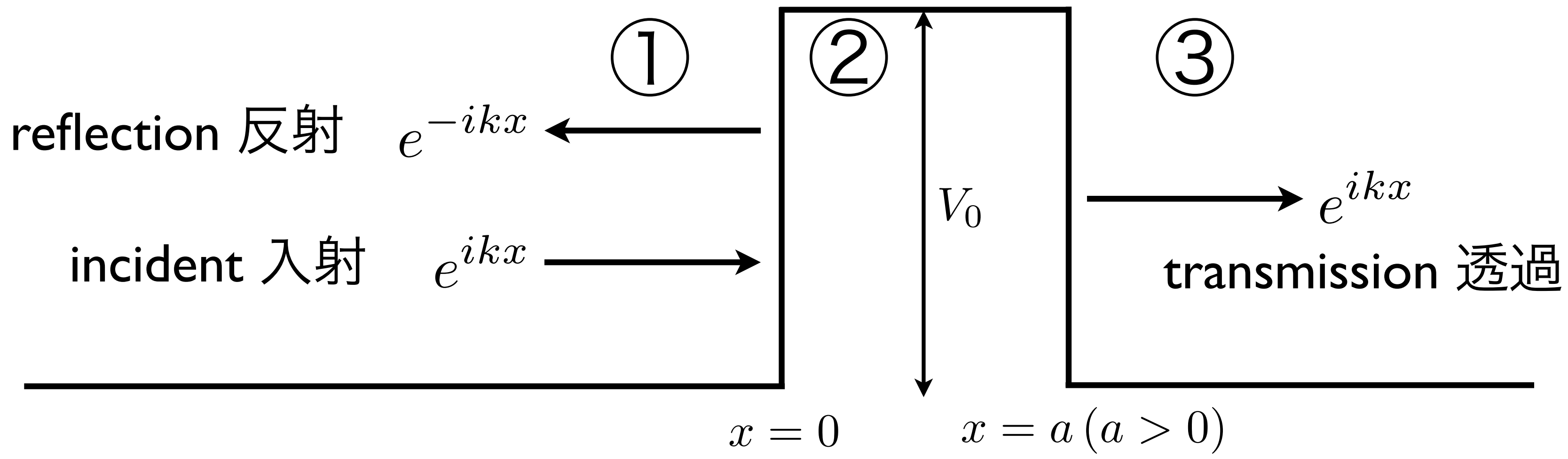
$$k = \sqrt{\frac{2mE}{\hbar^2}}$$



general solution $\psi(x) = A\,e^{ikx} + B\,e^{-ikx}$

$$\Psi(x,t) = A\,e^{i(kx-\omega t)} + B\,e^{i(-kx-\omega t)}$$

Barrier potential



$$E = \frac{\hbar^2 k^2}{2m} < V_0$$ 100 % reflection in classical mechanics 古典力学では100%反射される

$$\psi_1 = e^{ikx} + Ae^{-ikx} \qquad \psi_2 = Be^{Kx} + Ce^{-Kx} \qquad \psi_3 = De^{ikx}$$

$$K = \sqrt{2m(V_0 - E)/\hbar^2}$$

transmission coefficient 透過係数

$\psi, \frac{d\psi}{dx}$ continuous ➡

$$T = |D|^2 = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 Ka}$$

$$E = \frac{\hbar^2 k^2}{2m} < V_0$$

100 % reflection in classical mechanics　古典力学では100%反射される

transmission coefficient　透過係数

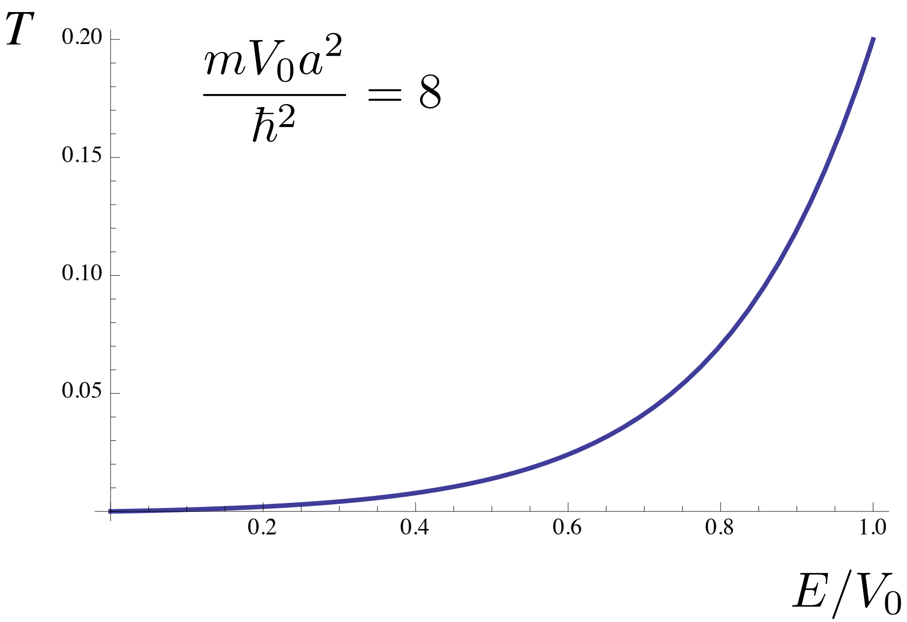
$$T = |D|^2 = \frac{1}{1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 Ka}$$

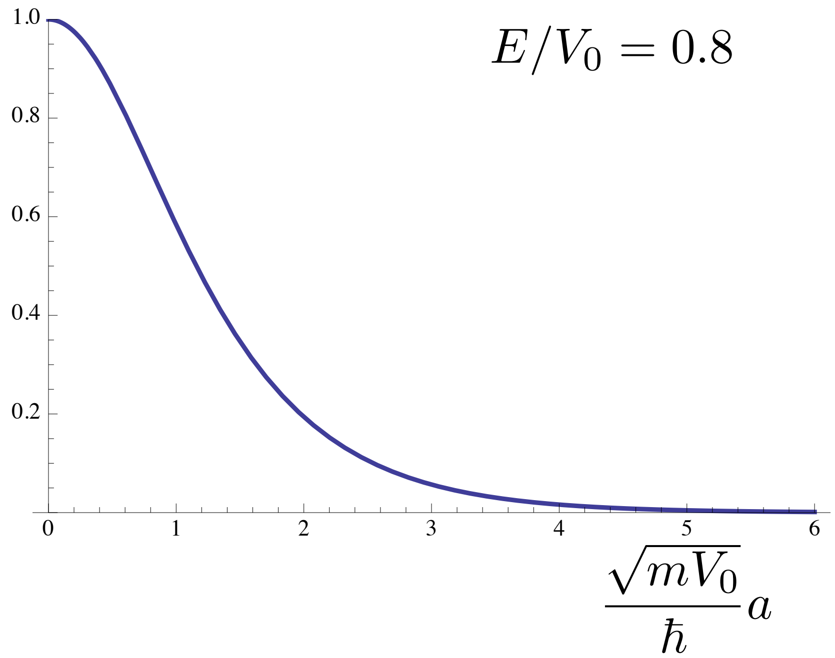
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

transmission is nonzero in quantum mechanics → **tunneling effect**　トンネル効果

量子力学では透過がある

important in alpha decay





Parity

パリティ（偶奇性）

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi(x) = E\,\psi(x)$$

if $V(x) = V(-x)$ $\bar{\psi}(x) = \psi(-x)$ **satisfies**

$$-\frac{\hbar^2}{2m}\frac{d^2\bar{\psi}}{dx^2} + V(x)\,\bar{\psi}(x) = E\,\bar{\psi}(x)$$

in the absence of degeneracy 縮退がなければ

$$\bar{\psi}(x) = P\psi(x) \qquad |P| = 1$$

$$\psi(x) = P^2\psi(x) \quad \longrightarrow \quad P = \pm 1$$

P = 1	$\psi(-x) = \psi(x)$	even or + parity	偶（+）のパリティ
P = -1	$\psi(-x) = -\psi(x)$	odd or - parity	奇（–）のパリティ

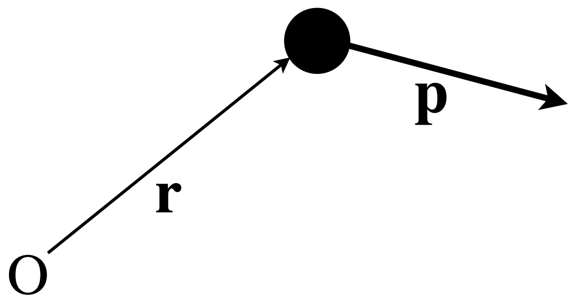
Nuclear states can be assigned a definite parity, even or odd.

Important in the discussion of beta decay

Angular momentum

角運動量

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

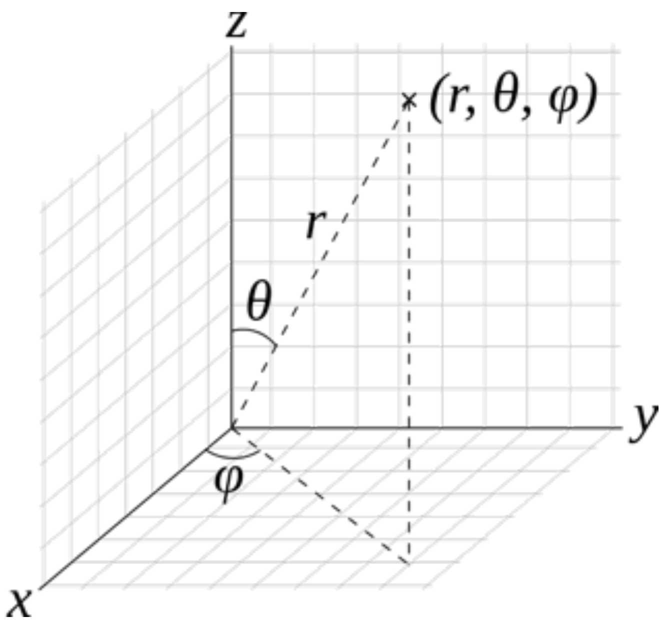


$$L_x = -i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right)$$

$$L_y = -i\hbar \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right)$$

$$L_z = -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$$

In the spherical coordinate　極座標では



$$L_x = i\hbar(\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\tan\theta}\frac{\partial}{\partial\phi})$$

$$L_y = i\hbar(-\cos\phi\frac{\partial}{\partial\theta} + \frac{\sin\phi}{\tan\theta}\frac{\partial}{\partial\phi})$$

$$L_z = -i\hbar\frac{\partial}{\partial\phi}$$

$$\mathbf{L}^2 = -\hbar^2 \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right]$$

commutation relation 交換関係

$$[\mathbf{L}^2, L_x] = [\mathbf{L}^2, L_y] = [\mathbf{L}^2, L_z] = 0$$

$$[L_x, L_y] = i\hbar L_z \qquad [L_y, L_z] = i\hbar L_x \qquad [L_z, L_x] = i\hbar L_y$$

common eigenfunction of $\mathbf{L}^2$ and L_z ➡ $Y_{lm}(\theta, \phi)$

固有関数

spherical harmonics
球面調和関数

$$\mathbf{L}^2\, Y_{lm}(\theta, \phi) = \hbar^2 l(l+1) Y_{lm}(\theta, \phi) \qquad l = 0, 1, 2, 3, \cdots$$

$$l_z\, Y_{lm}(\theta, \phi) = m\hbar Y_{lm}(\theta, \phi) \qquad m = -l, -l+1, \cdots, l-1, l$$

$(2l+1)$ eigenfunctions for given l
$(2l+1)$ 個の固有状態

examples

$$Y_{00} = \sqrt{\frac{1}{4\pi}} \qquad Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta \qquad Y_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$$

how about spin?

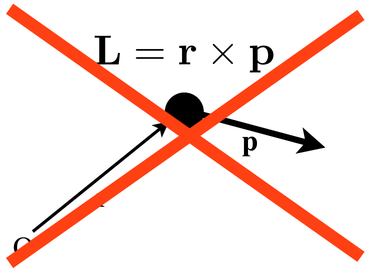
Let's use the commutation relation

$$[L_x, L_y] = i\hbar L_z \quad [L_y, L_z] = i\hbar L_x \quad [L_z, L_x] = i\hbar L_y$$

as a **definition** of *angular momentum (operator)*

$$\mathbf{L} = (L_x, L_y, L_z)$$

forget



For given l, there are $(2l+1)$ eigenfunctions (eigenvectors)
固有ベクトル

→ $(2l+1) \times (2l+1)$ matrix
行列

$$l = 1$$

$$L_z \to \hbar \begin{pmatrix} l & & & & & \\ & l-1 & & & & \\ & & l-2 & & & \\ & & & \cdots & & \\ & & & & -l+1 & \\ & & & & & -l \end{pmatrix}$$

$$\hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$L_+ = L_x + iL_y \to \hbar \begin{pmatrix} 0 & \sqrt{1 \cdot 2l} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{2 \cdot (2l-1)} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \sqrt{3 \cdot (2l-2)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \sqrt{2l \cdot 1} \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$\hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$$

$$L_- = L_x - iL_y \to \hbar \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \sqrt{2l \cdot 1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{(2l-1) \cdot 2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{(2l-2) \cdot 3} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \sqrt{1 \cdot 2l} & 0 \end{pmatrix}$$

$$\hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

Let's consider 2×2 matrices?

$$L_z \to \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad L_+ \to \frac{\hbar}{2}\begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix} \qquad L_- \to \frac{\hbar}{2}\begin{pmatrix}0 & 0\\ 2 & 0\end{pmatrix}$$

$$L_x \to \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad L_y \to \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$

eigenvalues = $\pm\dfrac{\hbar}{2}$ ⟺ spin $\frac{1}{2}$ particle

$$\mathbf{L}^2 \to \hbar^2 \frac{1}{2}\left(\frac{1}{2}+1\right)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad l = \frac{1}{2}$$

Pauli matrices パウリ行列

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \qquad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$